

Dispersionless integrable hierarchies and $GL(2, \mathbb{R})$ geometry

E.V. Ferapontov, B. Kruglikov

Department of Mathematical Sciences
Loughborough University
Loughborough, Leicestershire LE11 3TU
United Kingdom,

Institute of Mathematics and Statistics
Faculty of Science
University of Tromsø
Tromsø 90-37, Norway

e-mails:

`E.V.Ferapontov@lboro.ac.uk`

`Boris.Kruglikov@uit.no`

Abstract

Paraconformal or $GL(2, \mathbb{R})$ geometry on an n -dimensional manifold M is defined by a field of rational normal curves of degree $n - 1$ in the projectivised cotangent bundle $\mathbb{P}T^*M$. Such geometry is known to arise on solution spaces of ODEs with vanishing Wünschmann (Doubrov-Wilczynski) invariants. In this paper we discuss yet another natural source of $GL(2, \mathbb{R})$ structures, namely integrable hierarchies of the dispersionless Kadomtsev-Petviashvili (dKP) type. In the latter context, $GL(2, \mathbb{R})$ structures coincide with the characteristic variety (principal symbol) of the hierarchy.

Dispersionless hierarchies provide explicit examples of various particularly interesting classes of $GL(2, \mathbb{R})$ structures studied in the literature. Thus, we obtain torsion-free $GL(2, \mathbb{R})$ structures of Bryant [5] that appeared in the context of exotic holonomy in dimension four, as well as totally geodesic $GL(2, \mathbb{R})$ structures of Krynski [29]. The latter, also known as involutive $GL(2, \mathbb{R})$ structures, possess a compatible affine connection (with torsion) and a two-parameter family of totally geodesic α -manifolds (coming from the dispersionless Lax equations), which makes them a natural generalisation of the Einstein-Weyl geometry. As our main result we demonstrate that involutive $GL(2, \mathbb{R})$ structures are governed by a dispersionless integrable system, thereby establishing integrability of the Wünschmann conditions.

MSC: 35Q51, 37K10, 37K25, 53A40, 53B05, 53B50, 53C26, 53C80.

Keywords: $GL(2, \mathbb{R})$ Geometry, Dispersionless Integrable Hierarchy, Characteristic Variety, Compatible Connection, Lax Representation.

Contents

1	Introduction	2
1.1	$GL(2, \mathbb{R})$ geometry	2
1.2	Involutive $GL(2, \mathbb{R})$ structures	3
1.3	Dispersionless hierarchies and involutive $GL(2, \mathbb{R})$ structures	4
1.4	Connections associated with involutive $GL(2, \mathbb{R})$ structures	5
1.5	Summary of the main results	7
2	Examples of involutive $GL(2, \mathbb{R})$ structures	8
2.1	$GL(2, \mathbb{R})$ structures via dKP hierarchy	8
2.2	$GL(2, \mathbb{R})$ structures via the universal hierarchy	9
2.3	$GL(2, \mathbb{R})$ structures via Adler-Shabat triples	10
3	General involutive $GL(2, \mathbb{R})$ structures	11
3.1	Parametrisation of involutive $GL(2, \mathbb{R})$ structures	11
3.2	Integrability of involutive $GL(2, \mathbb{R})$ structures	14
3.3	Counting α -manifolds	15
3.4	Equivalent definitions of involutivity	16
4	Concluding remarks	17
5	Appendix: canonical connections	17
5.1	Connections associated with the dKP hierarchy	17
5.2	Connections associated with the universal hierarchy	18
5.3	Connections associated with Adler-Shabat triples	19

1 Introduction

1.1 $GL(2, \mathbb{R})$ geometry

On an n -dimensional manifold M , a $GL(2, \mathbb{R})$ geometry (also known as paraconformal geometry [12], or a rational normal structure [7], or the special case of a cone structure [20]) is defined by a field of rational normal curves of degree $n - 1$ in the projectivised cotangent bundle $\mathbb{P}T^*M$. Equivalently, it can be viewed as a field of 1-forms $\omega(\lambda)$ polynomial of degree $n - 1$ in λ ,

$$\omega(\lambda) = \omega_0 + \lambda\omega_1 + \cdots + \lambda^{n-1}\omega_{n-1}, \quad (1)$$

where ω_i is a basis of 1-forms (a coframe) on M . The parameter λ and the 1-form $\omega(\lambda)$ are defined up to transformations $\lambda \rightarrow \frac{a\lambda+b}{c\lambda+d}$, $\omega(\lambda) \rightarrow r(c\lambda+d)^{n-1}\omega(\lambda)$, where a, b, c, d, r are arbitrary smooth functions on M .

Conventionally, a $GL(2, \mathbb{R})$ geometry is defined by a field of rational normal curves in the projectivised *tangent* bundle $\mathbb{P}TM$. Our choice of the cotangent bundle is motivated by the fact that characteristic varieties of PDEs, which will be our main source of $GL(2, \mathbb{R})$ structures, are subvarieties of $\mathbb{P}T^*M$. In any case, both pictures are projectively dual: the equation $\omega(\lambda) = 0$ defines a one-parameter family of hyperplanes that osculate a dual rational normal curve $\tilde{\omega}(\lambda) \subset \mathbb{P}TM$. Below we discuss some of the most natural occurrences of $GL(2, \mathbb{R})$ structures.

Poisson geometry: Given a generic pair of compatible Poisson bivectors η_1, η_2 of Kronecker type on an odd-dimensional manifold N^{2k+1} , there is a canonical $GL(2, \mathbb{R})$ structure on the

base M^{k+1} (leaf space) of the corresponding action foliation (see [42]). As shown by Gelfand and Zakharevich [19] such structures, also known as Veronese webs, arise in the theory of bi-Hamiltonian integrable systems.

Exotic holonomy: It was observed by Bryant in [5] that, in four dimensions, there exist torsion-free affine connections whose holonomy group is the irreducible representation of $GL(2, \mathbb{R})$. Such connections give rise to canonically defined parallel $GL(2, \mathbb{R})$ structures. Historically, this was the first example of an ‘exotic’ holonomy not appearing on the Berger list [3], we refer to [6, 35] for the development of the holonomy problem.

Submanifolds in Grassmannians: Let M be a submanifold of the Grassmannian $Gr(k, n)$. The flat Segre structure of $Gr(k, n)$ induces on M a generalised conformal structure. Particular instances of this construction result in a $GL(2, \mathbb{R})$ geometry on M .

Thus, let M^4 be a fourfold in the Grassmannian $Gr(3, 5)$. The flat Segre structure of $Gr(3, 5)$ induces a field of twisted cubics on $\mathbb{P}TM^4$, that is, a $GL(2, \mathbb{R})$ structure on M^4 . These structures were investigated in [11] in the context of integrability in Grassmann geometries.

Similarly, let $\Lambda(3)$ be the Grassmannian of 3-dimensional Lagrangian subspaces of a 6-dimensional symplectic space. Given a hypersurface $M^5 \subset \Lambda(3)$, the flat Veronese structure of $\Lambda(3)$ induces a $GL(2, \mathbb{R})$ structure on M^5 . Such structures were discussed in [17, 38] in the context of integrability of dispersionless Hirota type equations.

Algebraic geometry: Given a compact complex surface X and a rational curve $C \subset X$ with the normal bundle $\nu \simeq \mathcal{O}(n)$, the results of Kodaira [31] show that there is a complex-analytic $(n+1)$ -dimensional moduli space M consisting of deformations of C , which carries a canonical $GL(2, \mathbb{R})$ structure. This was studied in detail by Hitchin for $n = 2$ (using É. Cartan’s work on Einstein-Weyl geometry) and by Bryant for $n = 3$. The construction generalises to the case when X is a holomorphic contact 3-fold and $C \subset X$ is a contact rational curve with the normal bundle $\nu \simeq \mathcal{O}(n-1) \oplus \mathcal{O}(n-1)$ [5, 7].

Ordinary differential equations: For every scalar (higher order) ODE with vanishing Wünschmann (Doubrov-Wilczynski) invariants, the space M of its solutions is canonically endowed with a $GL(2, \mathbb{R})$ structure. ODEs of this type have been thoroughly investigated in the literature, see e.g. [12, 10, 36, 21, 14, 29] and references therein.

Dispersionless integrable hierarchies: Given a dispersionless integrable hierarchy of dKP type, it will be demonstrated in this paper that the corresponding characteristic variety (zero locus of the principal symbol) determines canonically a $GL(2, \mathbb{R})$ structure on every solution. In a somewhat different language, examples of this type appeared in [13, 29], although the observation that these structures coincide with the characteristic variety is apparently new. We will show that the $GL(2, \mathbb{R})$ structures appearing on solutions to integrable hierarchies are not arbitrary, and must satisfy an important property of *involutivity*.

1.2 Involutive $GL(2, \mathbb{R})$ structures

For every $x \in M$, the equation $\omega(\lambda) = 0$ defines a 1-parameter family of hyperplanes in $T_x M$ parametrised by λ ; these are known as α -hyperplanes. A codimension one submanifold of M is said to be an α -manifold if all its tangent spaces are α -hyperplanes [29]. A $GL(2, \mathbb{R})$ structure is said to be *involutive* [7] or α -integrable [29] if every α -hyperplane is tangential to some α -manifold (one can show that α -manifolds of an involutive $GL(2, \mathbb{R})$ structure depend on 1 arbitrary function of 1 variable; see Section 3.3). The existence of α -manifolds suggests that involutive $GL(2, \mathbb{R})$ structures are amenable to twistor-theoretic methods, cf. [20].

In particular, the above-mentioned $GL(2, \mathbb{R})$ structures that arise on solution spaces of ODEs with vanishing Wünschmann invariants are involutive. It was shown in [29] that conversely, every involutive (α -integrable) $GL(2, \mathbb{R})$ structure can be obtained from an ODE of this type. Four-dimensional involutive $GL(2, \mathbb{R})$ structures were extensively studied in [5] in the context of exotic holonomy. These investigations were developed further in [12, 10, 36, 21, 14]. We will relate different approaches to involutivity in Section 3.4.

In Section 3 we give a local parametrisation of general involutive $GL(2, \mathbb{R})$ structures in terms of solutions of a certain dispersionless integrable system.

1.3 Dispersionless hierarchies and involutive $GL(2, \mathbb{R})$ structures

Our main observation is that involutive $GL(2, \mathbb{R})$ structures are induced, as characteristic varieties, on solutions to various dispersionless integrable hierarchies. Moreover, α -manifolds come from integral manifolds of the associated dispersionless Lax equations. The following example is based on [42, 14, 29]. Equations of the Veronese web hierarchy have the form

$$(c_i - c_j)u_k u_{ij} + (c_j - c_k)u_i u_{jk} + (c_k - c_i)u_j u_{ik} = 0, \quad (2)$$

one equation for every triple (i, j, k) of distinct indices. Here u is a function on the n -dimensional manifold M with local coordinates x^1, \dots, x^n , coefficients c_i are pairwise distinct constants, and $u_i = u_{x^i}$ denote partial derivatives. The characteristic variety of system (2) is defined by a system of quadrics,

$$(c_i - c_j)u_k p_i p_j + (c_j - c_k)u_i p_j p_k + (c_k - c_i)u_j p_i p_k = 0,$$

which specify a rational normal curve in $\mathbb{P}T^*M$ parametrised as $p_i = \frac{u_i}{\lambda - c_i}$ (the ideal of a rational normal curve is generated by quadrics, see e.g. [22]). Explicitly,

$$\omega(\lambda) = p_i dx^i = \sum \frac{u_i}{\lambda - c_i} dx^i; \quad (3)$$

note that expression (3) takes form (1) on clearing the denominators (since only the conformal class of $\omega(\lambda)$ is essential we will not make a distinction in what follows). This supplies M with a $GL(2, \mathbb{R})$ geometry, which depends on the solution u (otherwise said: $GL(2, \mathbb{R})$ geometry on the solution u considered as a submanifold $\text{graph}(u) \subset M \times \mathbb{R}$).

System (2) is equivalent to the commutativity conditions of the following vector fields ($\lambda = \text{const}$),

$$\partial_{x^j} - \frac{\lambda - c_1}{\lambda - c_j} \frac{u_j}{u_1} \partial_{x^1}, \quad 1 < j \leq n,$$

which constitute dispersionless Lax representation for system (2). Note that these vector fields are annihilated by $\omega(\lambda)$. Their integral manifolds supply M with a two-parameter family of α -manifolds. Thus, $GL(2, \mathbb{R})$ structure (3) is involutive. Equivalently, the commutativity of these vector fields can be interpreted as the involutivity of the corresponding corank 2 vector distribution on the $(n+1)$ -dimensional manifold \hat{M} with coordinates x^1, \dots, x^n, λ , known as the correspondence space. The (complexified) space of integral manifolds of this distribution plays an important role in the twistor-theoretic approach to the Veronese web hierarchy.

In Section 2 we provide further examples of involutive $GL(2, \mathbb{R})$ structures supported on solutions to other well-known dispersionless integrable hierarchies.

1.4 Connections associated with involutive $GL(2, \mathbb{R})$ structures

There are several types of canonical connections that can be naturally associated with a $GL(2, \mathbb{R})$ structure on a manifold M . Recall that an affine connection ∇ is said to be compatible with a $GL(2, \mathbb{R})$ structure (paraconformal or $GL(2, \mathbb{R})$ -connection), if [29] for every $v \in TM$

$$\nabla_v \omega(\lambda) \in \text{span}\langle \omega(\lambda), \omega'(\lambda) \rangle, \quad (4)$$

where prime denotes differentiation by λ . Condition (4) means that the parallel transport defined by ∇ preserves rational normal cones of the $GL(2, \mathbb{R})$ structure. Equivalently, identifying quadratic equations from the ideal of the rational normal curve $\omega(\lambda)$ with symmetric bivectors g_s on M and denoting $g = \text{span}\langle g_s \rangle$, we can represent (4) as $\nabla_v g = g \ \forall v \neq 0$.

Condition (4) alone does not specify ∇ uniquely: for this, additional constraints should be imposed. In what follows, we discuss four types of canonical connections associated with involutive $GL(2, \mathbb{R})$ structures. We use the convention $\nabla_j \partial_k = \Gamma_{jk}^i \partial_i$.

Torsion-free $GL(2, \mathbb{R})$ connection

Torsion-free $GL(2, \mathbb{R})$ connections can only exist in four dimensions. Indeed, based on the Berger criteria, it was shown in [5] that there exist no non-trivial torsion-free $GL(2, \mathbb{R})$ connections in higher dimensions. On the contrary, in four dimensions, involutivity of a $GL(2, \mathbb{R})$ structure is equivalent to the existence of a torsion-free $GL(2, \mathbb{R})$ connection.

Since $GL(2, \mathbb{R})$ structures coming from principal symbols of dispersionless integrable hierarchies are automatically involutive (due to the existence of a Lax representation), we obtain an abundance of explicit examples of torsion-free $GL(2, \mathbb{R})$ connections in four dimensions parametrised by solutions to some well-known integrable PDEs, see Section 2.

For the Veronese web hierarchy, the Christoffel symbols of the torsion-free $GL(2, \mathbb{R})$ connection associated with four-dimensional $GL(2, \mathbb{R})$ structure (3) are computed to be equal to

$$\begin{aligned} \Gamma_{ii}^i &= \frac{u_{ii}}{u_i} - \frac{1}{9} \sum_{j \neq i} \frac{(c_{ik}c_{jl} + c_{il}c_{jk})^2}{c_{ik}c_{il}c_{jk}c_{jl}} \frac{u_{ij}}{u_j}, & \Gamma_{ii}^j &= \frac{1}{9} \frac{c_{jk}c_{jl}}{c_{ij}c_{lk}} \frac{u_i}{u_j} \left(\frac{u_{ik}}{u_k} - \frac{u_{il}}{u_l} \right), \\ \Gamma_{ij}^i &= \frac{1}{3} \frac{u_{ij}}{u_i} - \frac{1}{9} \left(1 + \frac{c_{ik}c_{jl}}{c_{ij}c_{kl}} \right) \frac{u_{jl}}{u_l} - \frac{1}{9} \left(1 + \frac{c_{il}c_{jk}}{c_{ij}c_{lk}} \right) \frac{u_{jk}}{u_k}, & \Gamma_{ik}^j &= \frac{1}{9} \frac{c_{lj}}{c_{lk}} \frac{u_k}{u_j} \left(\frac{u_{ik}}{u_k} - \frac{u_{il}}{u_l} \right), \end{aligned}$$

here $c_{ij} = c_i - c_j$, and i, j, k, l are pairwise distinct indices taking values $1, \dots, 4$.

Totally geodesic and normal $GL(2, \mathbb{R})$ connections

A particularly interesting subclass of involutive $GL(2, \mathbb{R})$ structures was introduced by Krynski in [29]: such structures possess a $GL(2, \mathbb{R})$ connection (with torsion) and a two-parameter family of totally geodesic α -manifolds. We will refer to such structures/connections as *totally geodesic $GL(2, \mathbb{R})$ structures/connections*, respectively.

Remark 1. The requirement that ∇ is a totally geodesic $GL(2, \mathbb{R})$ connection specifies ∇ uniquely up to transformations of the form $\Gamma_{jk}^i \rightarrow \Gamma_{jk}^i + \phi_j \delta_k^i$, for a covector ϕ . This freedom can be eliminated by requiring that the torsion T_∇ is trace-free, $T_{ik}^i = 0$. In what follows this will be included into the totally geodesic condition. For $GL(2, \mathbb{R})$ structures (3) coming from the Veronese web hierarchy, the condition $\text{tr}(T_\nabla(\cdot, X)) = 0$ is equivalent to the constraint $T_\nabla(\tilde{\omega}(\lambda), \tilde{\omega}'(\lambda)) \in \text{span}\langle \tilde{\omega}(\lambda) \rangle$ used in [29].

Examples of totally geodesic $GL(2, \mathbb{R})$ structures include the following:

- *Four-dimensional* $GL(2, \mathbb{R})$ structures arising, as characteristic varieties, on solutions to various integrable hierarchies of dKP type (see Appendix). We emphasise that this is a merely 4-dimensional phenomenon, thus, 5-dimensional $GL(2, \mathbb{R})$ structures associated with the dKP hierarchy do not possess totally geodesic $GL(2, \mathbb{R})$ connections.
- *Multi-dimensional* $GL(2, \mathbb{R})$ structures arising, as characteristic varieties, on solutions to *linearly degenerate* integrable hierarchies (those having no ∂_λ in the Lax fields, such as the Veronese web hierarchy and the ‘universal’ hierarchy).

It was shown in [29] that totally geodesic $GL(2, \mathbb{R})$ connections ∇ satisfy the following multi-dimensional *generalized Einstein-Weyl property*. Namely, the symmetrised Ricci tensor of such ∇ belongs to the span \tilde{g} of symmetric bivectors defining the dual rational normal curve $\tilde{\omega}(\lambda)$: $Ric_{\nabla}^{\text{sym}} \in \tilde{g}$. Note that in 3D this is precisely the classical Einstein-Weyl condition.

It was questioned in [29] whether the generalized Einstein-Weyl property implies in turn the totally geodesic property. The answer to this is negative in general, so a weaker condition is required. Let us call a $GL(2, \mathbb{R})$ connection ∇ *normal* if its torsion T_{∇} satisfies the following properties:

- (i) T_{∇} is trace-free: $\text{tr}(T_{\nabla}(\cdot, X)) = 0 \quad \forall X$;
- (ii) T_{∇} preserves α -hyperplanes as a $(2,1)$ -map: $X, Y \in \omega(\lambda)^{\perp} \Rightarrow T_{\nabla}(X, Y) \in \omega(\lambda)^{\perp}$.

Every totally geodesic $GL(2, \mathbb{R})$ connection is necessarily normal, although the converse is not true in general. It turns out that for all hierarchies we investigated, the normal $GL(2, \mathbb{R})$ connection exists, and is unique (note that there are no totally geodesic connections associated with higher-dimensional $GL(2, \mathbb{R})$ structures coming from the dKP and the Adler-Shabat hierarchies, starting from dimension 5).

The importance of normal $GL(2, \mathbb{R})$ connections lies in the fact that every such ∇ satisfies the generalized Einstein-Weyl property, which also reflects integrability of the hierarchy.

Remark 2. It was shown in [5] that in four dimensions every $GL(2, \mathbb{R})$ structure, not necessarily involutive, possesses a canonical affine connection whose torsion lies in the irreducible $\mathfrak{sl}(2)$ -component \mathbb{V}_7 in the space of algebraic torsions, according to the decomposition

$$\Lambda^2(T^*M) \otimes TM = (\mathbb{V}_0 \oplus \mathbb{V}_4) \otimes \mathbb{V}_3 = \mathbb{V}_1 \oplus 2\mathbb{V}_3 \oplus \mathbb{V}_5 \oplus \mathbb{V}_7,$$

where \mathbb{V}_l denotes the standard irreducible $\mathfrak{sl}(2)$ -module of dimension $l + 1$. We emphasize that in all 4-dimensional examples of totally geodesic $GL(2, \mathbb{R})$ structures known to us, the corresponding trace-free torsion lies in \mathbb{V}_1 .

The totally geodesic (and thus normal) $GL(2, \mathbb{R})$ connection associated with $GL(2, \mathbb{R})$ structure (3) of the Veronese web hierarchy is given by the formula

$$\nabla_j \partial_k = \left(\frac{u_{jk}}{u_k} + \phi_j \right) \partial_k, \quad \text{or} \quad \Gamma_{jk}^i = \left(\frac{u_{jk}}{u_k} + \phi_j \right) \delta_k^i;$$

here the covector ϕ_j is still arbitrary [29]. It can be fixed uniquely by requiring the torsion to be trace-free:

$$\phi_j = -\frac{1}{n-1} \sum_{k \neq j} \frac{u_{jk}}{u_k}.$$

A canonical projective connection

There exists yet another class of connections associated with involutive $GL(2, \mathbb{R})$ structures, namely, torsion-free connections possessing a two-parameter family of totally geodesic α -manifolds; note that they do not preserve the $GL(2, \mathbb{R})$ structure in general.

For $GL(2, \mathbb{R})$ structures defined by the characteristic varieties of dispersionless hierarchies, the two-parameter family of totally geodesic α -manifolds comes from integral manifolds of the corresponding dispersionless Lax equations.

The requirement that ∇ is a torsion-free connection with totally geodesic α -manifolds specifies it uniquely up to projective equivalence, $\Gamma_{jk}^i \rightarrow \Gamma_{jk}^i + \phi_j \delta_k^i + \phi_k \delta_j^i$, for a 1-form ϕ . Thus, we obtain a canonically defined totally geodesic *projective* connection.

For the involutive $GL(2, \mathbb{R})$ structure (3) of the Veronese web hierarchy, the connection obtained by this recipe is computed to be equal to

$$\nabla_j \partial_k = \frac{u_{jk}}{2} \left(\frac{\partial_j}{u_j} + \frac{\partial_k}{u_k} \right).$$

On every solution, geodesics of this projective connection (considered as unparametrized curves) can be obtained by intersecting $n - 2$ generic α -manifolds.

1.5 Summary of the main results

In Section 2 we provide further explicit examples of involutive $GL(2, \mathbb{R})$ structures given by characteristic varieties of various dispersionless integrable hierarchies, namely the dKP hierarchy, the ‘universal’ hierarchy of Martinez-Alonso and Shabat, and the consistent Adler-Shabat triples. In each case we calculate the Christoffel symbols of the canonical connections discussed in Section 1.4 (these results are relegated to the Appendix).

In Section 3 we demonstrate that the general involutive $GL(2, \mathbb{R})$ structure can be brought to the normal form

$$\omega(\lambda) = \sum_{i=1}^n \frac{u_i}{\lambda - \frac{u_i}{v_i}} dx^i, \quad (5)$$

where the functions u and v satisfy a system of second-order PDEs,

$$\mathfrak{S}_{(jkl)} (a_i - a_j)(a_k - a_l) \left(\frac{2u_{ij} - (a_i + a_j)v_{ij}}{u_i u_j} + \frac{2u_{kl} - (a_k + a_l)v_{kl}}{u_k u_l} \right) = 0, \quad (6)$$

$$\mathfrak{S}_{(jkl)} (b_i - b_j)(b_k - b_l) \left(\frac{2v_{ij} - (b_i + b_j)u_{ij}}{v_i v_j} + \frac{2v_{kl} - (b_k + b_l)u_{kl}}{v_k v_l} \right) = 0, \quad (7)$$

where $a_i = \frac{u_i}{v_i}$, $b_i = \frac{v_i}{u_i}$, and \mathfrak{S} denotes cyclic summation over the indicated indices. We demonstrate that this overdetermined system is in involution, and its characteristic variety is the tangential variety of the rational normal curve $\omega(\lambda)$. Since the degree of the tangential variety equals $2n - 4$, we conclude that general involutive $GL(2, \mathbb{R})$ structures (modulo diffeomorphisms) depend on $2n - 4$ arbitrary functions of 3 variables. For $n = 4$ this reproduces the count of Bryant [5]. For general n , the functional freedom of $2n - 4$ arbitrary functions of 3 variables was also announced by Bryant in a series of talks in the early 2000s [7], but the proof was never published (we thank Robert Bryant for drawing out attention to these results). We also refer to [30] for an alternative PDE system governing involutive $GL(2, \mathbb{R})$ structures for $n = 4$.

In Section 3 we prove our main result:

Theorem 1 *System (6), (7) governing general involutive $GL(2, \mathbb{R})$ structures possesses a dispersionless Lax representation, and can be viewed as a dispersionless integrable hierarchy.*

It was shown in [29] that involutive $GL(2, \mathbb{R})$ structures are in one-to-one correspondence with ODEs having vanishing Wünschmann invariants. Thus, integrability of system (6), (7) implies integrability of the Wünschmann conditions.

2 Examples of involutive $GL(2, \mathbb{R})$ structures

In this section we give further examples of involutive $GL(2, \mathbb{R})$ structures arising on solutions of various dispersionless integrable hierarchies. Our main observation is that $GL(2, \mathbb{R})$ structures discussed in a similar context by Dunajski and Krynski in [13, 29] are nothing but characteristic varieties of the corresponding PDEs. This makes the construction entirely explicit and intrinsic.

We mainly focus on $GL(2, \mathbb{R})$ geometry in four dimensions, defined by the first three equations of the corresponding hierarchies. Higher dimensional generalisations are then obtained by adding higher flows (with higher time variables). The Christoffel symbols of the canonical connections associated with these examples are presented in the Appendix.

2.1 $GL(2, \mathbb{R})$ structures via dKP hierarchy

The first three equations of the dKP hierarchy have the form

$$\begin{aligned} u_{xt} - u_{yy} - u_x u_{xx} &= 0, \\ u_{xz} - u_{yt} - u_x u_{xy} - u_y u_{xx} &= 0, \\ u_{yz} - u_{tt} + u_x^2 u_{xx} - u_y u_{xy} &= 0. \end{aligned} \tag{8}$$

Here u is a function on the 4-dimensional manifold M with local coordinates x, y, t, z . The characteristic variety of this system is the intersection of three quadrics,

$$\begin{aligned} p_x p_t - p_y^2 - u_x p_x^2 &= 0, \\ p_x p_z - p_y p_t - u_x p_x p_y - u_y p_x^2 &= 0, \\ p_y p_z - p_t^2 + u_x^2 p_x^2 - u_y p_x p_y &= 0, \end{aligned} \tag{9}$$

which specify a rational normal curve (twisted cubic) in $\mathbb{P}T^*M$ parametrised as $p_x = 1$, $p_y = \lambda$, $p_t = \lambda^2 + u_x$, $p_z = \lambda^3 + 2u_x \lambda + u_y$, so that

$$\omega(\lambda) = dx + \lambda dy + (\lambda^2 + u_x)dt + (\lambda^3 + 2u_x \lambda + u_y)dz.$$

This supplies M with a $GL(2, \mathbb{R})$ geometry, which depends on the solution u . The occurrence of a rational normal curve in the theory of dKP hierarchy was also noted in [24] in the context of coisotropic deformations of algebraic curves.

Equations (8) are equivalent to the commutativity conditions of the following vector fields,

$$\begin{aligned} &\partial_y - \lambda \partial_x + u_{xx} \partial_\lambda, \\ &\partial_t - (\lambda^2 + u_x) \partial_x + (\lambda u_{xx} + u_{xy}) \partial_\lambda, \\ &\partial_z - (\lambda^3 + 2u_x \lambda + u_y) \partial_x + (\lambda^2 u_{xx} + \lambda u_{xy} + u_{xt} + u_x u_{xx}) \partial_\lambda, \end{aligned} \tag{10}$$

which constitute a dispersionless Lax representation. These vector fields live in the extended 5-dimensional space \hat{M} with coordinates x, y, t, z, λ ; note the explicit presence of ∂_λ . Projecting

integral manifolds of these vector fields from \hat{M} to M we obtain a two-parameter family of α -manifolds of the corresponding $GL(2, \mathbb{R})$ structure, thus establishing its involutivity.

Higher-dimensional generalisations of this construction can be obtained by taking higher flows of the dKP hierarchy. Thus, adding one extra variable τ (higher time), we obtain a system of 6 equations consisting of 3 equations (8) along with the following 3 equations added,

$$\begin{aligned} u_{x\tau} - u_{yz} - u_x u_{yy} - u_y u_{xy} - (u_t + u_x^2) u_{xx} &= 0, \\ u_{y\tau} - u_{tz} + u_x u_y u_{xx} + (u_x^2 - u_t) u_{xy} - u_y u_{yy} &= 0, \\ u_{t\tau} - u_{zz} + (u_y^2 - u_x^3 - u_x u_t) u_{xx} + 2u_x u_y u_{xy} - u_t u_{yy} &= 0, \end{aligned} \quad (11)$$

here u is a function on the 5-dimensional manifold M with local coordinates x, y, t, z, τ . The characteristic variety of this system is the intersection of 6 quadrics, obtained by adding to 3 quadrics (9) the following 3 quadratic equations,

$$\begin{aligned} p_x p_\tau - p_y p_z - u_x p_y^2 - u_y p_x p_y - (u_t + u_x^2) p_x^2 &= 0, \\ p_y p_\tau - p_t p_z + u_x u_y p_x^2 + (u_x^2 - u_t) p_x p_y - u_y p_y^2 &= 0, \\ p_t p_\tau - p_z^2 + (u_y^2 - u_x^3 - u_x u_t) p_x^2 + 2u_x u_y p_x p_y - u_t p_y^2 &= 0. \end{aligned}$$

Altogether, they specify a rational normal curve in $\mathbb{P}T^*M$ parametrised as $p_x = 1$, $p_y = \lambda$, $p_t = \lambda^2 + u_x$, $p_z = \lambda^3 + 2u_x \lambda + u_y$, $p_\tau = \lambda^4 + 3u_x \lambda^2 + 2u_y \lambda + u_t + u_x^2$, so that

$$\omega(\lambda) = dx + \lambda dy + (\lambda^2 + u_x) dt + (\lambda^3 + 2u_x \lambda + u_y) dz + (\lambda^4 + 3u_x \lambda^2 + 2u_y \lambda + u_t + u_x^2) d\tau.$$

Equations (8, 11) are equivalent to the commutativity conditions of the 4 vector fields, obtained by adding to 3 vector fields (10) the 4th field,

$$\begin{aligned} \partial_\tau - (\lambda^4 + 3u_x \lambda^2 + 2u_y \lambda + u_t + u_x^2) \partial_x \\ + (\lambda^3 u_{xx} + u_{xy} \lambda^2 + (u_{xt} + 2u_x u_{xx}) \lambda + u_{xz} + u_y u_{xx} + u_x u_{xy}) \partial_\lambda. \end{aligned}$$

These fields live in the extended 6-dimensional space \hat{M} with coordinates $x, y, t, z, \tau, \lambda$. Projecting their integral manifolds from \hat{M} to M we obtain a two-parameter family of α -manifolds of the corresponding $GL(2, \mathbb{R})$ structure.

2.2 $GL(2, \mathbb{R})$ structures via the universal hierarchy

The first three equations of the universal hierarchy of Martinez-Alonso and Shabat [34] have the form

$$\begin{aligned} u_{xt} - u_{yy} + u_y u_{xx} - u_x u_{xy} &= 0, \\ u_{xz} - u_{yt} + u_t u_{xx} - u_x u_{xt} &= 0, \\ u_{yz} - u_{tt} + u_t u_{xy} - u_y u_{xt} &= 0. \end{aligned} \quad (12)$$

Here u is a function on the 4-dimensional manifold M with local coordinates x, y, t, z . The characteristic variety of this system is the intersection of three quadrics,

$$\begin{aligned} p_x p_t - p_y^2 + u_y p_x^2 - u_x p_x p_y &= 0, \\ p_x p_z - p_y p_t + u_t p_x^2 - u_x p_x p_t &= 0, \\ p_y p_z - p_t^2 + u_t p_x p_y - u_y p_x p_t &= 0, \end{aligned}$$

which specify a rational normal curve in $\mathbb{P}T^*M$ parametrised as $p_x = 1$, $p_y = \lambda - u_x$, $p_t = \lambda^2 - u_x \lambda - u_y$, $p_z = \lambda^3 - u_x \lambda^2 - u_y \lambda - u_t$, so that

$$\omega(\lambda) = dx + (\lambda - u_x) dy + (\lambda^2 - u_x \lambda - u_y) dt + (\lambda^3 - u_x \lambda^2 - u_y \lambda - u_t) dz.$$

Equations (12) are equivalent to the commutativity conditions of the following vector fields,

$$\begin{aligned} & \partial_y - (\lambda - u_x)\partial_x, \\ & \partial_t - (\lambda^2 - u_x\lambda - u_y)\partial_x, \\ & \partial_z - (\lambda^3 - u_x\lambda^2 - u_y\lambda - u_t)\partial_x, \end{aligned}$$

in the extended space \hat{M} with coordinates x, y, t, z, λ . Note the absence of ∂_λ , which indicates a close similarity with the Veronese web hierarchy. Integral manifolds of these vector fields provide a two-parameter family of α -manifolds of the corresponding $GL(2, \mathbb{R})$ structure.

This has a straightforward higher dimensional generalisation: the equations are

$$u_{i,j+1} - u_{i+1,j} + u_j u_{1,i} - u_i u_{1,j} = 0, \quad 0 < i < j < n;$$

the $GL(2, \mathbb{R})$ structure is given by

$$\omega(\lambda) = \sum_{i=1}^n (\lambda^{i-1} - u_1 \lambda^{i-2} - \dots - u_{i-1}) dx^i;$$

the Lax representation is

$$X_i = \partial_{x^i} - (\lambda^{i-1} - u_1 \lambda^{i-2} - \dots - u_{i-1}) \partial_{x^1}, \quad 1 < i \leq n.$$

Considered altogether, these equations form an integrable hierarchy.

2.3 $GL(2, \mathbb{R})$ structures via Adler-Shabat triples

Further examples of $GL(2, \mathbb{R})$ structures arise as characteristic varieties on solutions to triples of consistent 3D second-order PDEs discussed by Adler and Shabat in [1],

$$\begin{aligned} u_{23} &= f(u_1, u_2, u_3, u_{12}, u_{13}), \\ u_{24} &= g(u_1, u_2, u_4, u_{12}, u_{14}), \\ u_{34} &= h(u_1, u_3, u_4, u_{13}, u_{14}), \end{aligned} \tag{13}$$

where u is a function on the 4-dimensional manifold M with local coordinates x^1, \dots, x^4 . Note that system (2) belongs to class (13). As yet another example of this type let us consider the system

$$u_{23} = \frac{u_{12} - u_{13}}{u_2 - u_3}, \quad u_{24} = \frac{u_{12} - u_{14}}{u_2 - u_4}, \quad u_{34} = \frac{u_{13} - u_{14}}{u_3 - u_4}. \tag{14}$$

Its characteristic variety is defined by a system of quadrics,

$$p_2 p_3 = \frac{p_1 p_2 - p_1 p_3}{u_2 - u_3}, \quad p_2 p_4 = \frac{p_1 p_2 - p_1 p_4}{u_2 - u_4}, \quad p_3 p_4 = \frac{p_1 p_3 - p_1 p_4}{u_3 - u_4},$$

which specify a rational normal curve in $\mathbb{P}T^*M$ parametrised as $p_1 = 1$, $p_i = \frac{1}{\lambda - u_i}$, so that

$$\omega(\lambda) = dx^1 + \frac{1}{\lambda - u_2} dx^2 + \frac{1}{\lambda - u_3} dx^3 + \frac{1}{\lambda - u_4} dx^4.$$

System (14) is equivalent to the conditions of commutativity of the following vector fields,

$$\partial_{x^2} + \frac{1}{u_2 - \lambda} \partial_{x^1} + \frac{u_{12}}{u_2 - \lambda} \partial_\lambda, \quad \partial_{x^3} + \frac{1}{u_3 - \lambda} \partial_{x^1} + \frac{u_{13}}{u_3 - \lambda} \partial_\lambda, \quad \partial_{x^4} + \frac{1}{u_4 - \lambda} \partial_{x^1} + \frac{u_{14}}{u_4 - \lambda} \partial_\lambda,$$

note the explicit presence of ∂_λ . Projecting their integral manifolds from the extended space \hat{M} to M we obtain a two-parameter family of α -manifolds of the corresponding $GL(2, \mathbb{R})$ structure.

This has a straightforward higher-dimensional generalization: the equations are

$$(u_i - u_j)u_{ij} = u_{1i} - u_{1j}, \quad 1 < i < j \leq n;$$

the $GL(2, \mathbb{R})$ structure is given by

$$\omega(\lambda) = dx^1 + \sum_{i=2}^n \frac{1}{\lambda - u_i} dx^i;$$

the Lax representation is

$$X_i = \partial_{x^i} - \frac{1}{\lambda - u_i} \partial_{x^1} - \frac{u_{1i}}{\lambda - u_i} \partial_\lambda, \quad 1 < i \leq n.$$

Considered altogether, these equations form an integrable hierarchy.

3 General involutive $GL(2, \mathbb{R})$ structures

In this section we demonstrate that general involutive $GL(2, \mathbb{R})$ structures are governed by a dispersionless integrable hierarchy. This will prove Theorem 1.

3.1 Parametrisation of involutive $GL(2, \mathbb{R})$ structures

We begin by encoding all involutive structures in one ansatz.

Proposition 1. *Every involutive $GL(2, \mathbb{R})$ structure can be represented in form (5),*

$$\omega(\lambda) = \sum_{i=1}^n \frac{u_i}{\lambda - \frac{u_i}{v_i}} dx^i,$$

where u and v are functions of $x = (x^1, \dots, x^n)$, and subscripts denote partial derivatives: $u_i = u_{x^i}$, $v_i = v_{x^i}$. The functions u and v are not arbitrary, and must satisfy a system of PDEs (6), (7) coming from the integrability condition $d\omega(\lambda) \wedge \omega(\lambda) = 0$.

Proof:

Let (1) be an involutive $GL(2, \mathbb{R})$ structure on an n -dimensional manifold M . It is easy to see that the space of α -manifolds is at least 2-dimensional (in fact, it is parametrised by 1 arbitrary function of 1 variable). Choosing a 1-parameter subfamily of α -manifolds we obtain a (local) foliation of M . This foliation consists of integral manifolds of an integrable distribution $\omega(a) = 0$ obtained by substituting λ by some function a on M . We can thus set $\omega(a) = f dx$ for some functions f and x . Taking n different 1-parameter subfamilies of α -manifolds corresponding to the choice of n functions a_i such that $\omega(a_i) = f_i dx^i$ (no summation), and using x^i as local coordinates on M , we can reduce $\omega(\lambda)$ to the form

$$\omega(\lambda) = \sum \frac{\varphi_i}{\lambda - a_i} dx^i.$$

Let us further require that $\omega(a_{n+1}) = g du$ and $\omega(a_{n+2}) = h dv$. Using the linear-fractional freedom, $\lambda \rightarrow \frac{m\lambda + n}{p\lambda + q}$, where m, n, p, q are arbitrary functions on M , we can reduce a_{n+1} and

a_{n+2} to ∞ and 0, respectively. This implies that one can set $\varphi_i = u_i$, $a_i = \frac{u_i}{v_i}$, which leads to the required form (5).

The integrability condition $d\omega(\lambda) \wedge \omega(\lambda) = 0$ imposes constraints on the first-order derivatives of λ (here λ is viewed as a function of x), as well as a system of second-order PDEs for u and v that govern involutive $GL(2, \mathbb{R})$ structures. Namely, for every quadruple of distinct indices i, j, k, l one has second-order quasilinear PDEs (6), (7), i.e. vanishing of

$$E_{ijkl} = \underset{(jkl)}{\mathfrak{S}} (a_i - a_j)(a_k - a_l) \left(\frac{2u_{ij} - (a_i + a_j)v_{ij}}{u_i u_j} + \frac{2u_{kl} - (a_k + a_l)v_{kl}}{u_k u_l} \right),$$

as well as of a similar expression obtained by interchanging u and v ,

$$F_{ijkl} = \underset{(jkl)}{\mathfrak{S}} (b_i - b_j)(b_k - b_l) \left(\frac{2v_{ij} - (b_i + b_j)u_{ij}}{v_i v_j} + \frac{2v_{kl} - (b_k + b_l)u_{kl}}{v_k v_l} \right),$$

recall that $a_i = \frac{u_i}{v_i}$, $b_i = \frac{v_i}{u_i}$. Although system (6), (7) formally consists of $2\binom{n}{4}$ equations, only $2\binom{n-2}{2}$ of them are linearly independent. Indeed, we can restrict to equations E_{12kl} and F_{12kl} for $3 \leq k < l \leq n$ since all other equations are their linear combinations: denoting $\alpha_{ij} = a_i - a_j$ we have

$$\alpha_{12}E_{ijkl} = \alpha_{kl}E_{12ij} + \alpha_{jl}E_{12ki} + \alpha_{jk}E_{12il} + \alpha_{il}E_{12jk} + \alpha_{ik}E_{12lj} + \alpha_{ij}E_{12kl}$$

for all indices distinct (note that $\alpha_{ij} \neq 0$ for $i \neq j$) and similarly for F_{ijkl} .

For $n = 4$ system (6), (7) consists of 2 second-order PDEs for u and v , with the general solution parametrised by 4 arbitrary functions of 3 variables. This can be seen as an explicit confirmation of the result of [5], namely that modulo diffeomorphisms general involutive $GL(2, \mathbb{R})$ structures in four dimensions depend on 4 functions of 3 variables. \square

Proposition 2. *For every value of n , the following holds:*

- (a) *The characteristic variety of system (6), (7) is the tangential variety of the rational normal curve (5); it has degree $2n - 4$. Rational normal curve (5) can be recovered as the singular locus of the characteristic variety.*
- (b) *System (6), (7) is in involution.*
- (c) *The general solution of system (6), (7) depends on $2n - 4$ functions of 3 variables.*

Proof:

(a) Let us parametrize rational normal curve (5) as

$$\lambda \mapsto [p_1 : \dots : p_n] \in \mathbb{P}T^*M, \quad p_i = \frac{u_i}{\lambda - a_i}, \quad a_i = \frac{u_i}{v_i}, \quad (15)$$

so that its tangential variety is given by

$$(\lambda, \mu) \mapsto [p_1 : \dots : p_n] \in \mathbb{P}T^*M, \quad p_i = \frac{u_i}{\lambda - a_i} + \frac{u_i \mu}{(\lambda - a_i)^2}. \quad (16)$$

Let $E = E[u, v]$ and $F = F[u, v]$ be non-linear differential operators on the left-hand sides of (6) and (7). The symbol of the system $\mathcal{E} = \{E = 0, F = 0\}$ is given by the matrix

$$M_{\mathcal{E}} = \begin{bmatrix} \ell_E^u(p) & \ell_E^v(p) \\ \ell_F^u(p) & \ell_F^v(p) \end{bmatrix},$$

where $\ell_E^u(p) = \sum_{a \leq b} \frac{\partial E}{\partial u_{ab}} p_a p_b$ is the symbol of u -linearization of E , etc. As noted above, $E = (E_{ijkl})$ has $\binom{n-2}{2}$ independent components, and similarly for $F = (F_{ijkl})$, so that the matrix $M_{\mathcal{E}}$ is of the size $2 \times 2\binom{n-2}{2}$. The characteristic variety is defined by the formula

$$\text{Char}(\mathcal{E}) = \{[p] \in \mathbb{P}T^*M : \text{rank}(M_{\mathcal{E}}) < 2\}.$$

From (6) we have

$$\ell_{E_{ijkl}}^u(p) = 2 \underset{(jkl)}{\mathfrak{S}} (a_i - a_j)(a_k - a_l) \left(\frac{p_i p_j}{u_i u_j} + \frac{p_k p_l}{u_k u_l} \right).$$

This expression vanishes if we substitute p from (15). Similarly, all other components $\ell_{E_{ijkl}}^v(p)$, $\ell_{F_{ijkl}}^u(p)$, $\ell_{F_{ijkl}}^v(p)$ of the symbolic matrix vanish, and we conclude that $M_{\mathcal{E}} = 0$ modulo (15).

For the tangential variety (16), the entries of $M_{\mathcal{E}}$ do not vanish identically, however, a straightforward computation shows that independently of $(ijkl)$ we get

$$\lambda \ell_{E_{ijkl}}^u(p) + \ell_{E_{ijkl}}^v(p) = 0 \quad \text{and} \quad \lambda \ell_{F_{ijkl}}^u(p) + \ell_{F_{ijkl}}^v(p) = 0.$$

Thus, all columns of $M_{\mathcal{E}}$ are proportional whenever p satisfies (16), and $\text{rank}(M_{\mathcal{E}}) = 1$ unless p belongs to the rational normal curve (in which case we have $\text{rank}(M_{\mathcal{E}}) = 0$). Finally, for a rational normal curve of degree $n - 1$, the degree of its tangential variety equals $2n - 4$, and it is generated by quartics [22].

(b) The involutivity of system (6), (7) is established as follows. Recall [25] that the fiber of the jet-bundle $J^\infty(M)$ over a point $o \in M$ is the space $\mathcal{R} = \mathbb{R}[p_1, \dots, p_n] = ST_o^*M = \bigoplus_{k=0}^\infty S^k T_o^*M$ of homogeneous polynomials on $T_o M$. The bundle $J^\infty(M, \mathbb{R}^2)$ has fiber $\mathcal{R}^2 = \mathcal{R} \otimes_{\mathbb{R}} \mathbb{R}^2$. Our system $\mathcal{E} = \{E_{12kl} = 0, F_{12kl} = 0\}$ is a submanifold of the latter, and its symbol is the map $\ell_{\mathcal{E}} : \mathcal{R}^2 \rightarrow \mathcal{R}^2 \otimes \Lambda^2 \mathcal{R}^{n-2}$ (here all tensor products are over \mathcal{R}). Examination of the symbol shows that it defines a determinantal ideal of the type that is resolved by doubling the Eagon-Northcott complex [15, Appendix A2] that we write in the dualized form (arrows inverted):

$$\mathcal{R}^2 \xrightarrow{\ell_{\mathcal{E}}} \mathcal{R}^2 \otimes \Lambda^2 \mathcal{R}^{n-2} \xrightarrow{\quad} \mathcal{R}^{*2} \otimes \mathcal{R}^2 \otimes \Lambda^3 \mathcal{R}^{n-2} \rightarrow S^2 \mathcal{R}^{*2} \otimes \mathcal{R}^2 \otimes \Lambda^4 \mathcal{R}^{n-2} \rightarrow \dots$$

This complex is exact and so is a resolution of the kernel of $\ell_{\mathcal{E}} = (\ell_E, \ell_F)$. It is also worth noting that the Eagon-Northcott complex arises in the proof of the fact that the ideal of a rational normal curve is Cohen-Macaulay [15, A2.19].

To derive compatibility conditions, we follow the approach of [26], where commutative algebra (the Buchsbaum-Rim complex) was applied to quantify the compatibility conditions of overdetermined systems of generalized intersection type. In our case, due to the form of the Eagon-Northcott differential δ [15], the map $\varsigma = \delta^*$ has components $\varsigma_\tau^\sigma = \sum a_\tau^{\sigma i} p_i$ linear in p . Here $\sigma = (\sigma', \sigma'')$ and the $\binom{n-2}{2}$ components of both σ' and σ'' encode $\Lambda^2 \mathcal{R}^{n-2}$, while τ is the index for $\text{Im } \varsigma$. Thus (for a fixed multi-index τ) the τ -component of the map ς is

$$(\ell_E(p), \ell_F(p)) \mapsto \sum a_\tau^{\sigma' i} p_i \ell_{E_{\sigma'}}(p) + a_\tau^{\sigma'' i} p_i \ell_{F_{\sigma''}}(p).$$

The corresponding compatibility condition (differential syzygy) is the following (D_i is the total derivative by x^i)

$$\sum a_\tau^{\sigma' i} D_i E_{\sigma'} + a_\tau^{\sigma'' i} D_i F_{\sigma''} = 0 \mod (E_{\sigma_1}, F_{\sigma_2}),$$

i.e. the left-hand side, which is a function of 2-jets, vanishes on \mathcal{E} for every τ .

One can verify that only 5-tuples of distinct indices enumerating the equations in our system are used in the compatibility conditions. In terms of the above complex this visualizes as follows: the factor $\Lambda^3 \mathcal{R}^{n-2}$ (the third component of τ) in the space of compatibility conditions refers to triples of indices (klm) that yield the equations $E_{12kl}, E_{12km}, E_{12lm}$ and $F_{12kl}, F_{12km}, F_{12lm}$. For each such 5-tuple $(12klm)$ the number of compatibility conditions is four, which equals the dimension (over \mathcal{R}) of the first factor $\mathcal{R}^{*2} \otimes \mathcal{R}^2$ in the third term of the complex.

Thus it suffices to check compatibility for $n = 5$ to conclude it for general n . For $n = 5$ our complex becomes a short exact sequence $\mathcal{R}^2 \xrightarrow{\ell_\xi} \mathcal{R}^6 \xrightarrow{\varsigma} \mathcal{R}^4$, from which we read off 4 compatibility conditions (with the symbol ς). A direct verification (using symbolic computations in Maple) shows that they are satisfied. This implies the involutivity.

(c) Since system (6), (7) is in involution, and its characteristic variety has degree $2n - 4$ and (affine) dimension 3, the formal theory of differential systems [8, 25] implies that the general solution depends on $2n - 4$ arbitrary functions of 3 variables. \square

Remark 3. The system \mathcal{E} can be represented in a simple parametric form,

$$\frac{2u_{ij} - (a_i + a_j)v_{ij}}{u_i u_j} = p_i + p_j + \sum_{s=3}^{n-1} l_s f_s(a_i, a_j), \quad \frac{2v_{ij} - (b_i + b_j)u_{ij}}{v_i v_j} = q_i + q_j + \sum_{s=3}^{n-1} m_s f_s(b_i, b_j),$$

$1 \leq i < j \leq n$, where $f_s(\alpha, \beta) = \frac{\alpha^s - \beta^s}{\alpha - \beta}$. This system has $n(n - 1)$ equations and $4n - 6$ parameters $p_1, \dots, p_n, q_1, \dots, q_n, l_3, \dots, l_{n-1}, m_3, \dots, m_{n-1}$. Elimination of these parameters yields $(n - 2)(n - 3)$ equations (6), (7).

3.2 Integrability of involutive $GL(2, \mathbb{R})$ structures

Proposition 3. *For every n , system (6), (7) possesses a dispersionless Lax representation, and so can be viewed as a dispersionless integrable system. Varying n , we obtain the corresponding dispersionless integrable hierarchy.*

Proof:

The corresponding Lax representation is provided by equations for λ , namely, for every triple of distinct indices i, j, k the condition $d\omega(\lambda) \wedge \omega(\lambda) = 0$ implies the following first-order relation,

$$\frac{\lambda - a_i}{u_i} \left(\frac{1}{\lambda - a_k} - \frac{1}{\lambda - a_j} \right) \lambda_i + \frac{\lambda - a_j}{u_j} \left(\frac{1}{\lambda - a_i} - \frac{1}{\lambda - a_k} \right) \lambda_j + \frac{\lambda - a_k}{u_k} \left(\frac{1}{\lambda - a_j} - \frac{1}{\lambda - a_i} \right) \lambda_k + S_{ijk} = 0, \quad (17)$$

where $\lambda_i = \lambda_{x^i}$, and

$$S_{ijk} = u_{ij} \frac{a_j - a_i}{u_i u_j} \left(\frac{\lambda}{\lambda - a_i} + \frac{\lambda}{\lambda - a_j} \right) + u_{ik} \frac{a_i - a_k}{u_i u_k} \left(\frac{\lambda}{\lambda - a_i} + \frac{\lambda}{\lambda - a_k} \right) + u_{jk} \frac{a_k - a_j}{u_j u_k} \left(\frac{\lambda}{\lambda - a_j} + \frac{\lambda}{\lambda - a_k} \right) \\ - v_{ij} \frac{a_j - a_i}{u_i u_j} \left(\frac{\lambda a_i}{\lambda - a_i} + \frac{\lambda a_j}{\lambda - a_j} \right) - v_{ik} \frac{a_i - a_k}{u_i u_k} \left(\frac{\lambda a_i}{\lambda - a_i} + \frac{\lambda a_k}{\lambda - a_k} \right) - v_{jk} \frac{a_k - a_j}{u_j u_k} \left(\frac{\lambda a_j}{\lambda - a_j} + \frac{\lambda a_k}{\lambda - a_k} \right).$$

Modulo (6), (7), system (17) contains $n - 2$ linearly independent relations. The compatibility of equations for λ is equivalent to the system \mathcal{E} . Introducing the associated vector fields

$$V_{ijk} = \frac{\lambda - a_i}{u_i} \left(\frac{1}{\lambda - a_k} - \frac{1}{\lambda - a_j} \right) \partial_{x^i} + \frac{\lambda - a_j}{u_j} \left(\frac{1}{\lambda - a_i} - \frac{1}{\lambda - a_k} \right) \partial_{x^j} + \frac{\lambda - a_k}{u_k} \left(\frac{1}{\lambda - a_j} - \frac{1}{\lambda - a_i} \right) \partial_{x^k} - S_{ijk} \partial_\lambda$$

in the extended space \hat{M} with coordinates x^1, \dots, x^n, λ , we obtain the distribution $V = \langle V_{ijk} \rangle$. Direct computation shows that, by virtue of equations (6), (7), the distribution V is integrable, and $\text{rank } V = n - 2$ (here λ is treated as an independent variable). This provides a dispersionless

Lax representation for the system \mathcal{E} . Projecting integral manifolds of V from \hat{M} to M we obtain a 3-parameter family of codimension 2 submanifolds of M . Tangent spaces to these submanifolds are $(n-2)$ -dimensional osculating spaces of the dual curve $\tilde{\omega}(\lambda)$. Indeed, the distribution V is annihilated by the (pulled-back) 1-forms $\omega(\lambda)$ and $\omega'(\lambda)$.

As n increases, the equations describing general involutive $GL(2, \mathbb{R})$ structures exhibit a nested structure. Thus the combination of two properties of the system \mathcal{E} , namely involutivity and integrability, results in the corresponding dispersionless integrable hierarchy. \square

In the context of the general heavenly equation, similar Lax equations appeared recently in [4]. A dispersionless modification of the inverse scattering transform for Lax equations in parameter-dependent vector fields was developed in [33].

Remark 4. System (6), (7) governing general involutive $GL(2, \mathbb{R})$ structures can be viewed as a generalisation of the Veronese web hierarchy. Indeed, the Veronese web hierarchy results upon setting $v_i = \frac{1}{c_i} p u_i$, where c_i are constants and p is some function. Then the reparametrisation $\lambda \rightarrow \lambda/p$ identifies $GL(2, \mathbb{R})$ structure (5) with (3) (up to unessential conformal factor p), so that system (6), (7) reduces to equations (2) of the Veronese web hierarchy.

Another class of (translationally non-invariant) integrable deformations of the Veronese web hierarchy was considered recently in [28]: the corresponding Lax equations do not however contain ∂_λ , and are specifically 3-dimensional.

Remark 5. For $n = 4$ there exists a unique torsion-free $GL(2, \mathbb{R})$ connection associated with $GL(2, \mathbb{R})$ structure (5). It can be parametrised as

$$\Gamma_{jk}^i = \frac{u_i^2 u_j u_k}{(a_i - a_j)(a_i - a_k)} \psi_i, \quad \Gamma_{jj}^i = \frac{u_i^2 u_j^2}{(a_i - a_j)^2} \psi_i, \quad \Gamma_{ij}^i = \Gamma_{ji}^i = \frac{u_i u_j}{a_i - a_j} \phi_i, \quad \Gamma_{ii}^i = \rho_i,$$

where $i, j, k \in \{1, \dots, 4\}$ are pairwise distinct indices, and the quantities ψ_i, ϕ_i, ρ_i are yet to be determined from the linear system

$$\begin{aligned} \frac{u_{ij}}{u_i u_j} - \sum_p \Gamma_{ij}^p \frac{u_p}{u_i u_j} &= s_2 a_i a_j + s_1 (a_i + a_j) + s_0, \\ \frac{v_{ij}}{v_i v_j} - \sum_p \Gamma_{ij}^p \frac{v_p}{v_i v_j} &= \tilde{s}_2 b_i b_j + \tilde{s}_1 (b_i + b_j) + \tilde{s}_0, \end{aligned} \tag{18}$$

where s_j, \tilde{s}_j are extra parameters (to be eliminated). System (18) contains 20 linear equations for the 18 unknowns $\psi_i, \phi_i, \rho_i, s_j, \tilde{s}_j$. These equations are consistent modulo (6), (7), and lead to a unique torsion-free $GL(2, \mathbb{R})$ connection (we skip the final formulae due to their complexity).

3.3 Counting α -manifolds

The dispersionless Lax representation provides a two-parametric family of α -manifolds. The totality of all α -manifolds is bigger.

Proposition 4. *For an involutive $GL(2, \mathbb{R})$ structure, its local α -manifolds are parametrized by 1 function of 1 variable.*

Proof:

Let us invoke a relation with ordinary differential equations having all Wünschmann invariants zero, see [29] for details (recall that all involutive structures arise on solution spaces of such ODEs). An ODE \mathcal{E} of order n is given by a submanifold $x_n = F(t, x_0, x_1, \dots, x_{n-1})$ in the jet-space $J^n = \mathbb{R}^{n+2}(t, x_0, \dots, x_n)$, and \mathcal{E} is diffeomorphic (via the jet-projection) to the

jet-space J^{n-1} . The solution space M^n is identified with the space of integral curves of the field $X_F = \partial_t + x_1\partial_0 + \dots + x_{n-1}\partial_{n-2} + F\partial_{n-1}$, where $\partial_i = \partial_{x_i}$ and $F = F(t, x_0, x_1, \dots, x_{n-1})$.

Denote by $\pi : J^{n-1} \rightarrow M = J^{n-1}/X_F$ the projection (since the construction is local, this quotient exists, and is non-singular), and let $\mathcal{D}_{n-1} = \langle \partial_1, \dots, \partial_{n-1} \rangle$ be the vertical distribution in J^{n-1} with respect to the projection of J^{n-1} to $J^0 = \mathbb{R}^2(t, x_0)$. The family of hyperplanes $\pi_*\mathcal{D}_{n-1} \subset TM$ parametrized by the coordinate $\lambda = t$ along integral curves of X_F coincides with α -hyperplanes of a $GL(2, \mathbb{R})$ structure on M provided the Wünschmann invariants vanish.

Thus α -manifolds are projections of integral manifolds of (maximal possible) dimension $n-1$ for the (non-holonomic) distribution

$$\mathcal{D}_n = \pi_*^{-1}\pi_*(\mathcal{D}_{n-1}) = \langle X_F, \partial_1, \dots, \partial_{n-1} \rangle = \langle \partial_t + x_1\partial_0, \partial_1, \dots, \partial_{n-1} \rangle.$$

This distribution has rank n and possesses a sub-distribution of Cauchy characteristics of rank $n-2$ given by $Ch(\mathcal{D}_n) = \langle \partial_2, \dots, \partial_{n-1} \rangle$. Consequently, integral manifolds of \mathcal{D}_n are foliated by the Cauchy characteristics, and therefore coincide with vertical lifts of Legendrian curves of the standard contact structure on the quotient $J^1 = J^{n-1}/Ch(\mathcal{D}_n)$.

Note that Legendrian curves are restored by their projection to the plane $J^0 = \mathbb{R}^2(t, x_0)$ (together with an initial point in J^1); the curves whose projections degenerate to a point correspond to the standard two-parameter family of α -manifolds. Since curves in the plane are parametrized by 1 function of 1 variable, the claim follows. \square

Remark 6. A section of the correspondence bundle $\lambda = \lambda(x)$ subject to the constraints $V_{ijk} \cdot \lambda = 0$ from the proof of Proposition 3 defines a foliation of M by α -manifolds $\omega(\lambda) = 0$. Since $\text{rank } V = n-2$, the general solution of this first-order system for λ depends on 1 function of 2 variables. This is in agreement with the claim that individual α -manifolds depend on 1 function of 1 variable (yielding an independent proof of this fact).

Remark 7. A PDE system with the general solution depending on 1 function of 1 variable has class $\omega = 1$ in the terminology of Sophus Lie. By his theorem [32, 27] any such system is solvable via ODEs. Thus even though 1-parameter families of α -manifolds are given by a Lax PDE, individual α -manifolds can be found by a simpler technique.

3.4 Equivalent definitions of involutivity

Here we demonstrate the equivalence of the definitions of involutivity in the sense of Bryant [7] (recalled below) and α -integrability in the sense of Krynski [29] (Section 1.2).

Consider a manifold M^n , the associated contact manifold $\mathbb{P}T^*M$ of dimension $2n-1$ with the contact distribution \mathcal{C}_M , and a submanifold $\mathcal{Z} \subset \mathbb{P}T^*M$ of dimension $n+1$ that corresponds to a $GL(2, \mathbb{R})$ structure on M . We have $\dim(T\mathcal{Z} \cap \mathcal{C}_M) = n$. Since the projection $\pi : \mathbb{P}T^*M \rightarrow M$ is surjective on \mathcal{Z} , the intersection $\mathcal{Z}_x = \mathcal{Z} \cap \pi^{-1}(x) \subset \mathbb{P}T_x^*M$ is a curve (rational normal curve) for each $x \in M$. For $p \in \mathcal{Z}_x$ we have $d_p\pi(T\mathcal{Z} \cap \mathcal{C}_M) = p^\perp \subset T_xM$, $p^\perp \simeq T\mathcal{Z} \cap \mathcal{C}_M/T_p\mathcal{Z}_x$.

Denote the contact form by ω . Then \mathcal{Z} is *involutive* in the sense of [7] if $\omega \wedge (d\omega)^2|_{\mathcal{Z}} = 0 \Leftrightarrow (d\omega|_{T\mathcal{Z} \cap \mathcal{C}_M})^2 = 0$ (and $d\omega|_{T\mathcal{Z} \cap \mathcal{C}_M} \neq 0$ for dimensional reasons), whence for the subbundle $\Pi_M = \text{Ker}(d\omega|_{T\mathcal{Z} \cap \mathcal{C}_M})$ we have $\text{rank } \Pi_M = n-2$. The local quotient $(S_M, D_M) = (\mathcal{Z}, T\mathcal{Z} \cap \mathcal{C}_M)/\Pi_M$ is a 3-dimensional contact manifold. Denoting the projection by $\rho : \mathcal{Z} \rightarrow S_M$, the corresponding α -manifolds can be represented in the form $\rho^{-1}(L)$, where $L \subset S_M$ is a Legendrian curve with respect to D_M (compare with the proof of Proposition 4 from Section 3.3).

Conversely, if for every $x \in M$, $p \in \mathcal{Z}_x$ there exists an α -manifold tangent to $p^\perp \subset T_xM$, then the restriction of the canonical conformally symplectic form $[d\omega]$ to $T\mathcal{Z} \cap \mathcal{C}_M$ has rank 2, and so involutivity in our sense (α -integrability) implies that in the sense of [7].

4 Concluding remarks

We conclude with two general comments.

- It was demonstrated that involutive $GL(2, \mathbb{R})$ structures in 4D or, equivalently, torsion-free connections with the irreducible $GL(2, \mathbb{R})$ holonomy, are governed by a dispersionless integrable system. It would be interesting to understand which special holonomies lead to nonlinear PDEs that are either explicitly solvable/linearisable, or belong to the class of dispersionless integrable systems.
- Interesting generalisations of involutive $GL(2, \mathbb{R})$ structures arise in the context of integrable hierarchies whose characteristic varieties are *elliptic curves*. For instance, the first two equations of the dispersionless Pfaff-Toda hierarchy [39] are of the form (see [2])

$$\begin{aligned} e^{F_{xx}} F_{xt} &= e^{F_{yy}} F_{yz}, \\ F_{zt} &= 2e^{F_{xx}+F_{yy}} \sinh(2F_{xy}). \end{aligned}$$

Here F is a function on the 4-dimensional manifold M with coordinates x, y, t, z . The characteristic variety of this system is a complete intersection of two quadrics in \mathbb{P}^3 :

$$\begin{aligned} e^{F_{xx}} p_x p_t + e^{F_{xx}} F_{xt} p_x^2 &= e^{F_{yy}} p_y p_z + e^{F_{yy}} F_{yz} p_y^2, \\ p_z p_t &= e^{F_{xx}+F_{yy}} (e^{2F_{xy}} (p_x + p_y)^2 - e^{-2F_{xy}} (p_x - p_y)^2). \end{aligned}$$

It specifies a field of elliptic curves in the projectivised cotangent bundle $\mathbb{P}T^*M$. This field will automatically be involutive. The geometry of such structures is yet unclear, primarily due to the lack of a naturally adapted connection (the corresponding elliptic curves have non-constant j -invariants).

5 Appendix: canonical connections

In this section we provide the Christoffel symbols of the canonical connections associated with 4D examples of Section 2. In all cases, totally geodesic α -manifolds are projections of integral manifolds of commuting vector fields from the dispersionless Lax representation. These computations together with the corresponding higher-dimensional (5D etc) counterparts (not included here) were performed in Maple's DIFFERENTIALGEOMETRY package.

5.1 Connections associated with the dKP hierarchy

We use the notation $(x^1, x^2, x^3, x^4) = (x, y, t, z)$, note that $\Gamma_{jk}^i \neq \Gamma_{kj}^i$ in general. Not listed Christoffel symbols are zero (unless the connection is torsion-free, in which case $\Gamma_{jk}^i = \Gamma_{kj}^i$).

Torsion-free $GL(2, \mathbb{R})$ connection is given by

$$\begin{aligned} \Gamma_{13}^1 &= u_{11}, \quad \Gamma_{14}^1 = u_{12}, \quad \Gamma_{14}^2 = 2u_{11}, \quad \Gamma_{22}^1 = \frac{4}{9}u_{11}, \quad \Gamma_{23}^1 = u_{12}, \quad \Gamma_{23}^2 = \frac{8}{9}u_{11}, \\ \Gamma_{24}^1 &= \frac{13}{9}u_{22} - \frac{4}{9}u_{13}, \quad \Gamma_{24}^2 = 2u_{12}, \quad \Gamma_{24}^3 = \frac{4}{3}u_{11}, \quad \Gamma_{33}^1 = \frac{11}{9}u_{13} - \frac{2}{9}u_{22}, \\ \Gamma_{33}^2 &= 2u_{12}, \quad \Gamma_{33}^3 = \frac{7}{9}u_{11}, \quad \Gamma_{34}^1 = 2u_{23} - u_{14} + \frac{4}{3}u_2 u_{11}, \quad \Gamma_{34}^2 = \frac{22}{9}u_{13} - \frac{4}{9}u_{22}, \\ \Gamma_{34}^3 &= 3u_{12}, \quad \Gamma_{34}^4 = \frac{2}{3}u_{11}, \quad \Gamma_{44}^1 = u_{33} + \frac{32}{9}u_1 u_{22} - \frac{41}{9}u_1 u_{13} - 2u_2 u_{12}, \\ \Gamma_{44}^2 &= 4u_{23} - 2u_{14} + 6u_2 u_{11}, \quad \Gamma_{44}^3 = \frac{16}{3}u_{13} - \frac{7}{3}u_{22}, \quad \Gamma_{44}^4 = 4u_{12}. \end{aligned}$$

Normal (totally geodesic) $GL(2, \mathbb{R})$ connection with trace-free torsion is given by

$$\begin{aligned}
\Gamma_{13}^1 &= u_{11}, \Gamma_{14}^1 = u_{12}, \Gamma_{14}^2 = 2u_{11}, \Gamma_{22}^1 = u_{11}, \Gamma_{23}^1 = u_{12}, \Gamma_{23}^2 = 2u_{11}, \\
\Gamma_{24}^1 &= 2u_{22} - u_{13}, \Gamma_{24}^2 = 2u_{12}, \Gamma_{24}^3 = 3u_{11}, \Gamma_{31}^1 = -\frac{2}{3}u_{11}, \Gamma_{32}^1 = u_{12}, \\
\Gamma_{32}^2 &= \frac{1}{3}u_{11}, \Gamma_{33}^1 = 2u_{22} - u_{13}, \Gamma_{33}^2 = 2u_{12}, \Gamma_{33}^3 = \frac{4}{3}u_{11}, \Gamma_{34}^2 = 4u_{22} - 2u_{13}, \\
\Gamma_{34}^1 &= 2u_{23} - u_{14} - 2u_2u_{11}, \Gamma_{34}^3 = 3u_{12}, \Gamma_{34}^4 = \frac{7}{3}u_{11}, \Gamma_{41}^1 = u_{12}, \Gamma_{41}^2 = -3u_{11}, \\
\Gamma_{42}^1 &= 4u_{13} - 3u_{22}, \Gamma_{42}^2 = 2u_{12}, \Gamma_{42}^3 = -2u_{11}, \Gamma_{43}^1 = 2u_{23} - u_{14} + 3u_2u_{11}, \\
\Gamma_{43}^2 &= 3u_{13} - u_{22}, \Gamma_{43}^3 = 3u_{12}, \Gamma_{43}^4 = -u_{11}, \Gamma_{44}^1 = 3u_{33} - u_1u_{13} - 2u_{24}, \\
\Gamma_{44}^2 &= 4u_{23} - 2u_{14} + u_2u_{11}, \Gamma_{44}^3 = 2u_{13} + u_{22}, \Gamma_{44}^4 = 4u_{12}.
\end{aligned}$$

Totally geodesic projective connection is given by

$$\begin{aligned}
\Gamma_{13}^1 &= -\frac{1}{2}u_{11}, \Gamma_{14}^1 = -u_{12}, \Gamma_{14}^2 = -\frac{1}{2}u_{11}, \Gamma_{22}^1 = u_{11}, \Gamma_{23}^1 = u_{12}, \Gamma_{23}^2 = \frac{1}{2}u_{11}, \\
\Gamma_{24}^1 &= \frac{3}{2}u_1u_{11} + u_{22}, \Gamma_{24}^3 = \frac{1}{2}u_{11}, \Gamma_{33}^1 = u_{22} - u_1u_{11}, \Gamma_{33}^2 = 2u_{12}, \\
\Gamma_{34}^1 &= u_{14} - 2u_1u_{12} - \frac{3}{2}u_2u_{11}, \Gamma_{34}^2 = \frac{1}{2}u_1u_{11} + 2u_{22}, \Gamma_{34}^3 = u_{12}, \\
\Gamma_{44}^1 &= 2u_{33} - u_1u_{22} - u_2u_{12} - u_{24}, \Gamma_{44}^2 = 2u_{14} - 4u_1u_{12} - 3u_2u_{11}, \Gamma_{44}^3 = 3u_{13} - u_1u_{11}.
\end{aligned}$$

5.2 Connections associated with the universal hierarchy

We again use the notation $(x^1, x^2, x^3, x^4) = (x, y, t, z)$, note that $\Gamma_{jk}^i \neq \Gamma_{kj}^i$ in general. Not listed Christoffel symbols are zero (unless the connection is torsion-free, in which case $\Gamma_{jk}^i = \Gamma_{kj}^i$).

Torsion-free $GL(2, \mathbb{R})$ connection is given by

$$\begin{aligned}
\Gamma_{11}^2 &= -\frac{2}{3}u_{11}, \Gamma_{11}^3 = -\frac{2}{3}u_1u_{11} - u_{12}, \Gamma_{21}^3 = -\frac{1}{3}u_{11}, \\
\Gamma_{11}^4 &= -\frac{1}{3}(2u_1^2 + u_2)u_{11} - u_1u_{12} - u_{13}, \Gamma_{21}^4 = -\frac{2}{3}u_1u_{11} - u_{12}, \\
\Gamma_{12}^2 &= -\frac{8}{9}u_{12}, \Gamma_{22}^2 = -\frac{4}{9}u_{11}, \Gamma_{12}^3 = \frac{1}{9}u_1u_{12} - \frac{5}{9}u_2u_{11} - u_{13}, \\
\Gamma_{22}^3 &= -\frac{2}{9}u_1u_{11} - \frac{7}{9}u_{12}, \Gamma_{32}^3 = -\frac{2}{9}u_{11}, \\
\Gamma_{12}^4 &= \frac{1}{9}(u_1^2 - 7u_2)u_{12} - \frac{1}{9}(7u_1u_2 + 3u_3)u_{11} - u_{14}, \\
\Gamma_{22}^4 &= \frac{1}{9}u_1u_{12} - \frac{5}{9}u_2u_{11} - u_{13}, \Gamma_{32}^4 = -\frac{4}{9}u_1u_{11} - \frac{2}{3}u_{12}, \\
\Gamma_{13}^3 &= \frac{1}{9}(u_1^2 + 4u_2)u_{12} - \frac{10}{9}u_3u_{11} - u_{14}, \Gamma_{23}^3 = -\frac{1}{9}(u_1^2 + 4u_2)u_{11} - u_{13}, \\
\Gamma_{33}^3 &= -\frac{2}{9}u_1u_{11} - \frac{5}{9}u_{12}, \Gamma_{43}^3 = -\frac{1}{9}u_{11}, \\
\Gamma_{13}^4 &= \frac{1}{9}(u_1^3 + 4u_1u_2)u_{12} - \frac{1}{9}(u_1^2u_2 + 5u_1u_3 + 4u_2^2)u_{11} - u_1u_{14} - \frac{1}{3}u_3u_{12} - u_{24}, \\
\Gamma_{23}^4 &= \frac{1}{9}(u_1^2 + 4u_2)u_{12} - \frac{2}{3}u_3u_{11} - u_{14}, \Gamma_{33}^4 = -\frac{1}{9}(2u_1^2 + 7u_2)u_{11} - \frac{1}{9}u_1u_{12} - u_{13}, \\
\Gamma_{43}^4 &= -\frac{2}{9}u_1u_{11} - \frac{1}{3}u_{12}, \Gamma_{14}^4 = \frac{1}{9}(u_1^4 + 5u_1^2u_2 + 6u_1u_3 + 4u_2^2)u_{12} - u_1u_{24} \\
&\quad - \frac{1}{9}(u_1^3u_2 + u_1^2u_3 + 4u_1u_2^2 + 12u_2u_3)u_{11} - (u_1^2 + u_2)u_{14} - u_{34}, \\
\Gamma_{24}^4 &= \frac{1}{9}(u_1^3 + 4u_1u_2 + 9u_3)u_{12} - \frac{1}{9}(u_1^2u_2 - 3u_1u_3 + 4u_2^2)u_{11} - u_1u_{14} - u_{24}, \\
\Gamma_{34}^4 &= -\frac{1}{9}u_1u_{22} - \frac{8}{9}u_1u_{13} + \frac{1}{3}u_2u_{12} - u_{23}, \Gamma_{44}^4 = -\frac{4}{9}u_1^2u_{11} - \frac{4}{3}u_1u_{12} - u_{22}.
\end{aligned}$$

Normal (totally geodesic) $GL(2, \mathbb{R})$ connection with trace-free torsion is given by

$$\begin{aligned}
\Gamma_{11}^2 &= \Gamma_{21}^3 = \Gamma_{31}^4 = -u_{11}, \quad \Gamma_{11}^3 = \Gamma_{21}^4 = -u_1 u_{11} - u_{12}, \\
\Gamma_{11}^4 &= -u_1^2 u_{11} - u_1 u_{12} - u_2 u_{11} - u_{13}, \quad \Gamma_{12}^1 = \Gamma_{22}^2 = \Gamma_{32}^3 = \Gamma_{42}^4 = -\frac{1}{3} u_{11}, \\
\Gamma_{12}^2 &= \Gamma_{22}^3 = \Gamma_{32}^4 = -u_{12}, \quad \Gamma_{12}^3 = \Gamma_{22}^4 = -u_2 u_{11} - u_{13}, \\
\Gamma_{12}^4 &= -u_1 u_2 u_{11} - u_2 u_{12} - u_3 u_{11} - u_{14}, \\
\Gamma_{13}^1 &= \Gamma_{23}^2 = \Gamma_{33}^3 = \Gamma_{43}^4 = -\frac{1}{3} u_1 u_{11} - \frac{2}{3} u_{12}, \quad \Gamma_{13}^2 = \Gamma_{23}^3 = \Gamma_{33}^4 = -u_{13}, \\
\Gamma_{13}^3 &= \Gamma_{23}^4 = -u_3 u_{11} - u_{14}, \quad \Gamma_{13}^4 = -u_1 u_3 u_{11} - u_1 u_{14} - u_2 u_{13} - u_{33}, \\
\Gamma_{14}^1 &= \Gamma_{24}^2 = \Gamma_{34}^3 = \Gamma_{44}^4 = -\frac{1}{3} u_1^2 u_{11} - \frac{1}{3} u_1 u_{12} - \frac{2}{3} u_2 u_{11} - u_{13}, \\
\Gamma_{14}^2 &= \Gamma_{24}^3 = \Gamma_{34}^4 = -u_{14}, \quad \Gamma_{14}^3 = \Gamma_{24}^4 = -u_1 u_{14} - u_{24}, \\
\Gamma_{14}^4 &= -u_1^2 u_{14} - u_1 u_{24} - u_2 u_{14} - u_{34},
\end{aligned}$$

note that $\Gamma_{ij}^k = \Gamma_{i+a,j}^{k+a}$, as long as the indices are in the range.

Totally geodesic projective connection is given by

$$\begin{aligned}
\Gamma_{12}^1 &= \Gamma_{13}^2 = \Gamma_{14}^3 = -\frac{1}{2} u_{11}, \quad \Gamma_{13}^1 = -\frac{1}{2} u_1 u_{11}, \quad \Gamma_{14}^1 = -\frac{1}{2} u_1^2 u_{11} - \frac{1}{2} u_1 u_{12} - \frac{1}{2} u_2 u_{11}, \\
\Gamma_{14}^2 &= -\frac{1}{2} u_1 u_{11} - \frac{1}{2} u_{12}, \quad \Gamma_{22}^1 = -u_{12}, \quad \Gamma_{23}^1 = -\frac{1}{2} u_2 u_{11} - u_{13}, \\
\Gamma_{24}^1 &= -\frac{1}{2} u_1 u_2 u_{11} - \frac{1}{2} u_2 u_{12} - \frac{1}{2} u_3 u_{11} - u_{14}, \quad \Gamma_{24}^2 = -\frac{1}{2} u_2 u_{11}, \\
\Gamma_{24}^3 &= -\frac{1}{2} u_{12}, \quad \Gamma_{33}^1 = -u_3 u_{11} - u_{14}, \quad \Gamma_{33}^2 = -u_{13}, \quad \Gamma_{33}^3 = u_{12}, \\
\Gamma_{34}^1 &= -\frac{1}{2} u_1 u_3 u_{11} - u_1 u_{14} - u_{24} - \frac{1}{2} u_3 u_{12}, \quad \Gamma_{34}^2 = -\frac{1}{2} u_3 u_{11} - u_{14}, \\
\Gamma_{34}^3 &= \frac{1}{2} u_{12}, \quad \Gamma_{44}^1 = -u_1^2 u_{14} - u_1 u_{24} - u_2 u_{14} - u_{34}, \\
\Gamma_{44}^2 &= -u_1 u_{14} - u_{24}, \quad \Gamma_{44}^3 = -u_{14}, \quad \Gamma_{44}^4 = u_{13}.
\end{aligned}$$

5.3 Connections associated with Adler-Shabat triples

In what follows, i, j, k are pairwise distinct indices taking values 2, 3, 4.

Torsion-free $GL(2, \mathbb{R})$ connection is given by (no summation unless specified):

$$\begin{aligned}
\Gamma_{11}^1 &= \frac{2}{9} (3R_d - R_a) - \frac{1}{9} \sum_{i \neq 1} \sigma_{ijk} u_{ii}, \quad \Gamma_{1i}^1 = \frac{1}{3} (\gamma_{jk} u_{ij} + \gamma_{kj} u_{ik}) - \frac{1}{9} R_e, \\
\Gamma_{11}^i &= \frac{1}{9} (u_i - u_j)(u_i - u_k)(\gamma_{jk}(u_{jj} - 4u_{ij}) + \gamma_{kj}(u_{kk} - 4u_{ik})), \\
\Gamma_{1i}^i &= -\frac{2}{9} R_a - \frac{4}{9} u_{ii} + \frac{1}{9} \frac{\gamma_{jk}}{\gamma_{ji}} (u_{jj} - u_{ij}) + \frac{1}{9} \frac{\gamma_{kj}}{\gamma_{ki}} (u_{kk} - u_{ik}) + \frac{5}{9} (u_{ij} + u_{ik}), \\
\Gamma_{1i}^j &= \frac{1}{9} \frac{\gamma_{ki}}{\gamma_{kj}} (u_{ii} - 4u_{ij} + 4u_{jk} - u_{kk}), \quad \Gamma_{ii}^1 = -\frac{1}{9} R_f, \quad \Gamma_{ij}^1 = -\frac{1}{9} R_f, \\
\Gamma_{ii}^i &= -\frac{1}{3} R_e + \frac{\sigma_{ijk} + 2}{9} (\gamma_{jk} u_{jj} + \gamma_{kj} u_{kk}) + \frac{2}{9} \left(\gamma_{jk} - \gamma_{ij} + \gamma_{ik} - \frac{\gamma_{ij} \gamma_{ik}}{\gamma_{jk}} \right) (u_{ij} - u_{ik}), \\
\Gamma_{ii}^j &= -\frac{\gamma_{ij}}{9} \frac{\gamma_{ki}}{\gamma_{kj}} (u_{ii} - 4u_{ij} + 4u_{jk} - u_{kk}), \quad \Gamma_{ij}^k = \frac{\gamma_{ij}}{9} (u_{ii} - 4u_{ik} + 4u_{jk} - u_{jj}), \\
\Gamma_{ij}^i &= -\frac{2}{3} \gamma_{ij} R_a - \frac{1}{9} (\gamma_{jk} - 4\gamma_{ij})(u_{jj} + u_{ik}) + \frac{1}{9} (\gamma_{jk} + 5\gamma_{ij})(u_{kk} + u_{ij}),
\end{aligned}$$

where

$$\gamma_{ij} = \frac{1}{u_i - u_j}, \quad \sigma_{ijk} = \frac{\gamma_{ij}}{\gamma_{ik}} + \frac{\gamma_{ik}}{\gamma_{ij}} = \frac{u_i - u_k}{u_i - u_j} + \frac{u_i - u_j}{u_i - u_k},$$

$$R_d = \sum_{i \neq 1} (\gamma_{ij} + \gamma_{ik}) u_{1i}, \quad R_e = \sum_{i \neq 1} (\gamma_{ij} + \gamma_{ik}) u_{ii}, \quad R_f = \sum_{i \neq 1} \gamma_{ij} \gamma_{ik} u_{ii}.$$

Normal (totally geodesic) $GL(2, \mathbb{R})$ connection with trace-free torsion is given by

$$\Gamma_{11}^1 = \frac{1}{3}(2R_b - R_a), \quad \Gamma_{1i}^1 = R_c, \quad \Gamma_{i1}^1 = -\frac{1}{3} \left(\frac{u_{ii} - u_{jj}}{u_i - u_j} + \frac{u_{ii} - u_{kk}}{u_i - u_k} + 5R_c \right),$$

$$\Gamma_{1i}^i = 2u_{kl} - \frac{1}{3}(R_a + R_b), \quad \Gamma_{i1}^i = -u_{ii} + \frac{u_i - u_k}{u_j - u_k} u_{ij} + \frac{u_i - u_j}{u_k - u_j} u_{ik},$$

$$\Gamma_{ii}^i = \Gamma_{i1}^1 + R_c, \quad \Gamma_{ij}^i = \frac{R_b - u_{ii} - 2u_{jk}}{u_i - u_j},$$

$$\Gamma_{ij}^j = \frac{2u_{ii} + u_{jj} - 3u_{ij} - u_{ik} + u_{jk}}{3(u_i - u_j)} - \frac{u_{ii} - u_{kk}}{3(u_i - u_k)},$$

where

$$R_a = u_{22} + u_{33} + u_{44}, \quad R_b = u_{23} + u_{24} + u_{34},$$

$$R_c = \frac{u_2 u_{34}}{(u_2 - u_3)(u_2 - u_4)} + \frac{u_3 u_{24}}{(u_3 - u_2)(u_3 - u_4)} + \frac{u_4 u_{23}}{(u_4 - u_2)(u_4 - u_3)}.$$

Totally geodesic projective connection is given by

$$\Gamma_{1i}^i = -\frac{1}{2}u_{ii}, \quad \Gamma_{ij}^i = \frac{2u_{ij} - u_{ii} - u_{jj}}{2(u_i - u_j)},$$

recall that $\Gamma_{jk}^i = \Gamma_{kj}^i$, all other Christoffel symbols are zero.

Acknowledgements

We thank L. Bogdanov, R. Bryant, B. Doubrov, M. Dunajski, W. Krynski and M. Pavlov for helpful discussions. BK acknowledges financial support from the LMS making this collaboration possible.

References

- [1] V.E. Adler, A.B. Shabat, *A model equation of the theory of solitons*, Theoret. and Math. Phys. **153**, no. 1 (2007) 1373-1387.
- [2] V. Akhmedova, A. Zabrodin, *Elliptic parameterization of Pfaff integrable hierarchies in the zero-dispersion limit*, Theoret. Mat. Fiz. **185**, no. 3 (2015) 410-422.
- [3] M. Berger, *Sur les groupes d'holonomie homogène des variétés à connexion affine et des variétés riemanniennes*, Bull. Soc. Math. France **83** (1955) 279-330.
- [4] L.V. Bogdanov, *Doubrov-Ferapontov general heavenly equation and the hyper-Kähler hierarchy*, J. Phys. A **48**, no. 23 (2015) 235202, 15 pp.

- [5] R.L. Bryant, *Two exotic holonomies in dimension four, path geometries, and twistor theory*, in: Complex geometry and Lie theory (Sundance, UT, 1989), 33–88, Proc. Sympos. Pure Math., 53, Amer. Math. Soc., Providence, RI (1991).
- [6] R. Bryant, *Classical, exceptional, and exotic holonomies: a status report*, Actes de la Table Ronde de Géométrie Différentielle (Luminy, 1992), 93165, Sémin. Congr., 1, Soc. Math. France, Paris, 1996.
- [7] R.L. Bryant, Involutive rational normal structures, <http://www.cim.nankai.edu.cn/activites/conferences/hy20090511/pdf/BryantConfSlides.pdf> (2009).
- [8] R.L. Bryant, S.S. Chern, R.B. Gardner, H.L. Goldschmidt, P.A. Griffiths, *Exterior differential systems*, MSRI Publications **18**, Springer-Verlag (1991).
- [9] D.M.J. Calderbank, *Integrable background geometries*, SIGMA **10** (2014), Paper 034, 51 pp; see also <http://people.bath.ac.uk/dmjc20/mpapers.html> (2002).
- [10] B. Doubrov, *Generalized Wilczynski invariants for non-linear ordinary differential equations*, Symmetries and overdetermined systems of partial differential equations, M. Eastwood, W. Miller (eds.), IMA Vol. Math. Appl. **144** Springer, NY (2008) 25–40.
- [11] B. Doubrov, E.V. Ferapontov, B. Kruglikov, V.S. Novikov, *On the integrability in Grassmann geometries: integrable systems associated with fourfolds in $Gr(3, 5)$* , arXiv:1503.02274.
- [12] M. Dunajski, P. Tod, *Paraconformal geometry of n th-order ODEs, and exotic holonomy in dimension four*, J. Geom. Phys. **56** (2006) 1790–1809.
- [13] M. Dunajski, Interpolating integrable system, arXiv:0804.1234v1.
- [14] M. Dunajski, W. Krynski, *Einstein-Weyl geometry, dispersionless Hirota equation and Veronese webs*, Math. Proc. Cambridge Philos. Soc. **157**, no. 1 (2014) 139–150.
- [15] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Springer, Berlin (1995).
- [16] E.V. Ferapontov, K.R. Khusnutdinova, *On the integrability of $(2+1)$ -dimensional quasilinear systems*, Comm. Math. Phys. **248** (2004) 187–206.
- [17] E.V. Ferapontov, L. Hadjikos, K.R. Khusnutdinova, *Integrable equations of the dispersionless Hirota type and hypersurfaces in the Lagrangian Grassmannian*, Int. Math. Res. Notices (2010) no. 3, 496–535.
- [18] E. V. Ferapontov, B. Kruglikov, *Dispersionless integrable systems in 3D and Einstein-Weyl geometry*, J. Diff. Geom. **97** (2014) 215–254.
- [19] I.M. Gelfand, I. Zakharevich, *Webs, Veronese curves, and bi-Hamiltonian systems*, J. Funct. Anal. **99**, no. 1 (1991) 150–178.
- [20] S.G. Gindikin, *Generalized conformal structures*, Twistors in mathematics and physics, 36–52, London Math. Soc. Lecture Note Ser. **156**, Cambridge Univ. Press, Cambridge (1990).
- [21] M. Godlinski, P. Nurowski, *$GL(2, R)$ geometry of ODE’s*, J. Geom. Phys. **60**, no. 6–8 (2010) 991–1027.
- [22] J. Harris, *Algebraic geometry: a first course*, Springer-Verlag, New York (1992), 328pp.
- [23] N.J. Hitchin, *Complex manifolds and Einstein’s equations*, Twistor geometry and nonlinear systems (Primorsko, 1980), 73–99, Lecture Notes in Math. **970**, Springer, Berlin-New York (1982).
- [24] B.G. Konopelchenko, G. Ortenzi, *Coisotropic deformations of algebraic varieties and integrable systems*, J. Phys. A **42**, no. 41 (2009) 415207, 18 pp.
- [25] B. Kruglikov, V. Lychagin, *Geometry of Differential equations*, Handbook of Global Analysis, Ed. D.Krupka, D.Saunders, Elsevier, (2008) 725–772.
- [26] B. Kruglikov, V. Lychagin, *Compatibility, Multi-brackets and Integrability of Systems of PDEs*, Acta Appl. Math. **109** (2010) 151–196.

- [27] B. Kruglikov, *Lie theorem via rank 2 distributions (integration of PDE of class $\omega = 1$)*, J. Nonlin. Math. Phys. **19**, no. 2 (2012) 1250011.
- [28] B. Kruglikov, A. Panasyuk, *Veronese webs and nonlinear PDEs*, Journal of Geometry and Physics, DOI: 10.1016/j.geomphys.2016.08.008; arXiv:1602.07346.
- [29] W. Krynski, *Paraconformal structures, ordinary differential equations and totally geodesic manifolds*, arXiv:1310.6855.
- [30] W. Krynski, T. Mettler, *$GL(2, \mathbb{R})$ -structures in dimension four, H -flatness and integrability*, arXiv:1611.08228.
- [31] K. Kodaira, *On stability of compact submanifolds of complex manifolds*, Amer. J. Math. **85** (1963) 79-94.
- [32] S. Lie, *Zur allgemeinen theorie der partiellen differentialgleichungen beliebiger ordnung*, Leipz. Berichte, Heft I, 53-128 (1895); Gesammelte Abhandlungen Bd.4, paper IX (Teubner and Aschehoug, Leipzig and Oslo, 1929).
- [33] S.V. Manakov, P.M. Santini, *The Cauchy problem on the plane for the dispersionless Kadomtsev-Petviashvili equation*, JETP Lett. **83** (2006) 462-6.
- [34] L. Martinez Alonso, A.B. Shabat, *Energy-dependent potentials revisited: a universal hierarchy of hydrodynamic type*, Phys. Lett. A **300** (2002) 58-54.
- [35] S. Merkulov, L. Schwachhöfer, *Classification of irreducible holonomies of torsion-free affine connections*, Ann. of Math. (2) **150**, no. 1 (1999) 77-149.
- [36] P. Nurowski, *Differential equations and conformal structures*, J. Geom. Phys. **55**, no. 1 (2005) 19-49.
- [37] P. Nurowski, *Comment on $GL(2, R)$ geometry of fourth-order ODEs*, J. Geom. Phys. **59**, no. 3 (2009) 267-278.
- [38] A.D. Smith, *Integrable $GL(2)$ geometry and hydrodynamic partial differential equations*, Communications in Analysis and Geometry **18**, no. 4 (2010) 743-790.
- [39] K. Takasaki, *Auxiliary linear problem, difference Fay identities and dispersionless limit of Pfaff-Toda hierarchy*, SIGMA Symmetry Integrability Geom. Methods Appl. **55** (2009), Paper 109, 34 pp.
- [40] P. Tod, *Einstein-Weyl spaces and third-order differential equations*, J. Math. Phys. **41**, no. 8 (2000) 5572-5581.
- [41] R.S. Ward, *Einstein-Weyl spaces and $SU(\infty)$ Toda fields*, Class. Quantum Grav. **7**, no. 4 (1990) L95-L98.
- [42] I. Zakharevich, *Nonlinear wave equation, nonlinear Riemann problem, and the twistor transform of Veronese webs*, arXiv:math-ph/0006001.